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THE SOLUTIONS OF A MODEL NONLINEAR SINGULAR PERTURBATION PROBLEM HAVING A CONTINUOUS LOCUS OF SINGULAR POINTS.

by

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THE SOLUTIONS OF A MODEL NONLINEAR SINGULAR PERTURBATION PROBLEM HAVING A CONTINUOUS LOCUS OF SINGULAR POINTS*

Gershon Kedem[1], Seymour V. Parter[2] and Michael Steuerwalt[3]

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ABSTRACT

y-square t-square

Consider the boundary value problem $\varepsilon y'' = (y^2 - t^2)y'$, $-1 \le t \le 0$, < or =
$y(-1) = A$, $y(0) = B$. We discuss the multiplicity of solutions and their limiting behavior as $\varepsilon \to 0+$ for certain choices of A and B. In
epsilon approaches
particular, when $A = 1$, $B = 0$ a bifurcation analysis gives a detailed and fairly complete analysis. The interest here arises from the complexity of the set of "turning points."

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THE SOLUTIONS OF A MODEL NONLINEAR SINGULAR PERTURBATION PROBLEM HAVING A CONTINUOUS LOCUS OF SINGULAR POINTS*

Gershon Kedem[(1)], Seymour V. Parter[(2)] and Michael Steuerwalt[(3)]

1. Introduction

Consider the nonlinear boundary value problem

1.1) $$\varepsilon y''(t) = (y^2 - t^2)y'(t), \qquad -1 \leq t \leq 0,$$

1.2) $$y(-1) = A, \qquad y(0) = B,$$

with $\varepsilon > 0$. It is not difficult to prove that there exists at least one solution, $y(t,\varepsilon)$. Moreover, if $y(t,\varepsilon)$ is a solution and $A \neq B$, then $|y'(t,\varepsilon)| > 0$.

The questions of interest are

(i) for $\varepsilon > 0$, how many solutions are there?

(ii) what are the "limit solutions," i.e. functions $Y(t)$ such that there are sequences $\varepsilon_n \to 0+$ and solutions $y(t,\varepsilon_n)$ so that

$$y(t,\varepsilon_n) \to Y(t), \quad \underline{\text{in some sense}} .$$

Again, it is well known that any such limit solution must satisfy the reduced equation.

1.3) $$(Y^2(t) - t^2)Y'(t) = 0 \qquad \text{a.e.}$$

In [3] F. Howes and S. V. Parter studied the case $Y(t) \equiv$ constant. Consider the case

1.4) $$0 \leq B < A \leq 1 .$$

Then, the only possible constant limit functions are (see [3])

1.5) $$Y(t) \equiv A, \quad Y(t) \equiv B, \quad Y(t) \equiv \frac{1}{\sqrt{3}} .$$

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In fact, computational results of Frances Sutton [10] imply that all three constant limit functions occur if possible, i.e. if

$$B < \frac{1}{\sqrt{3}} < A .$$

These computations of Sutton prompted G. Kedem [4,5] to apply a-posteriori estimates to this problem. He took the special case

$$\varepsilon = \frac{1}{15}, \quad B = .001, \quad A = .96$$

and proved the existence of at least three distinct solutions.

In this paper we continue to study these questions of multiplicity of solutions (for $\varepsilon > 0$) and limit solutions. In Section 2 we discuss some general facts about the solutions of this problem. In particular we discuss the occurrence of maximal and minimal solutions in the case where $A > B$.

In Section 3 we extend the results of [4,5] as follows: if $B = .001$, $A = .96$ and $0 < \varepsilon \leq \frac{1}{15}$, then there exist at least three solutions of equations (1.1), (1.2). In fact, there is a maximal solution $y_M(t,\varepsilon)$ and a minimal solution $y_m(t,\varepsilon)$ such that: if $y(t,\varepsilon)$ is any solution, then

1.6) $$y_m(t,\varepsilon) \leq y(t,\varepsilon) \leq y_M(t,\varepsilon) .$$

Moreover,

$$\lim y_M(t,\varepsilon) = A, \quad \varepsilon \to 0 ,$$
$$\lim y_m(t,\varepsilon) = B, \quad \varepsilon \to 0 .$$

Finally, there is a function $Y(t)$, not equal to A or B, and there is a sequence of solutions $y(t,\varepsilon_n)$ such that

$$y(t,\varepsilon_n) \to Y(t) \quad \text{as} \quad \varepsilon_n \to 0 .$$

In Section 4 we consider the special case $A = 1$, $B = 0$. In this case the function

1.7) $$y_0(t,\varepsilon) = -t$$

is a solution for all $\varepsilon > 0$. Thus we may apply results from bifurcation theory. In particular, we may apply a theorem of Paul Rabinowitz [7] to conclude that: if ε is small enough there are at least two distinct solutions $u_j(t,\varepsilon)$, $v_j(t,\varepsilon)$ which cross

$y_0(t,\varepsilon)$ exactly j times in the interior: $-1 < t < 0$. A further analysis of the limit behavior of these solutions then provides infinitely many step function limit functions.

In Section 5 we extend the analysis of Section 4 to the case $0 < A \neq 1$, $B = 0$. We obtain necessary and sufficient conditions for the existence of solutions with prescribed behavior near $t = -1$ having exactly j crossings of $y_0(t)$. The limit behavior as $\varepsilon \to 0+$ is discussed. Once more, the limit functions are step functions.

2. Maximal and Minimal Solutions

In this section we are concerned with the existence of maximal and minimal solutions $y_M(t,\varepsilon)$, $y_m(t,\varepsilon)$ described by (1.6). Let $\varepsilon > 0$ be fixed and let

2.1) $$U \equiv \{v(t) \in C[-1,0],\ \min(A,B) \leq y(t) \leq \max(A,B)\}\ .$$

Consider the mapping T which is defined by

2.2a) $$\varepsilon(Tv)'' = (v^2 - t^2)(Tv)'\ ,$$

2.2b) $$(Tv)(-1) = A,\quad (Tv)(0) = B\ .$$

Then, the maximum principle shows that $T(U) \subset U$. Moreover, for any $\bar{t} \in [-1,0]$

$$(Tv)'(t) = (Tv)'(\bar{t})\exp\{\frac{1}{\varepsilon}\int_{\bar{t}}^{t} (v^2(s) - s^2)ds\}\ .$$

Since

2.3a) $$\min|(Tv)'(s)| \leq |A - B|\ ,$$

2.3b) $$|v^2(s) - s^2| \leq \max(|A|^2, |B|^2) + 1$$

we have

2.4) $$|(Tv)'(t)| \leq |A - B|\exp\left\{\frac{\max(|A|^2, |B|^2) + 1}{\varepsilon}\right\}.$$

Thus, T is a compact mapping of U into itself. By the Schauder Fixed Point Theorem [2] there is at least one solution $y(t,\varepsilon)$.

Moreover, we make the following observations.

<u>Lemma 2.1</u>: Let $A > B \geq 0$. Let $v_0 \in U$. Suppose that

2.5a) $$Tv_0(t) \leq v_0(t)\ .$$

Let $v_j(t)$, $j = 1,2,\ldots$ be defined by

2.5b) $$v_j(t) = Tv_{j-1},\qquad j = 1,2,\ldots\ .$$

Then

2.5c) $$v_{j+1}(t) \leq v_j(t),\quad j = 1,2,\ldots\ .$$

Similarly, if

2.6a) $$Tv_0(t) \geq v_0(t)$$

and $v_j(t)$ is defined by (2.5b), then

2.6b) $$v_{j+1}(t) \geq v_j(t)\ .$$

Proof: We consider only the first case. We observe that $v_j'(t) \leq 0$ for $j = 1,2,\ldots$. Suppose that

$$0 \leq v_j(t) \leq v_{j-1}(t) .$$

Then

$$\varepsilon v_{j+1}'' = (v_j^2 - t^2)v_{j+1}' ,$$

$$\varepsilon v_j'' = (v_{j-1}^2 - t^2)v_j' .$$

Hence

2.7) $$\varepsilon(v_{j+1} - v_j)'' = (v_j^2 - t^2)(v_{j+1} - v_j)' + (v_j^2 - v_{j-1}^2)v_j' .$$

Since the second term on the right hand side of (2.7) is positive, the maximum principle asserts that

$$v_{j+1} - v_j \leq 0 .$$

Thus, the lemma is proven.

Lemma 2.2: Let $A > B \geq 0$. Let $v_0 \in U$ and suppose that (2.5a) holds. Let $y = y(t,\varepsilon)$ be a solution and suppose that

2.8) $$y(t,\varepsilon) \leq v_0(t) .$$

Then

2.9) $$y(t,\varepsilon) \leq v_j(t), \qquad j = 1,2,\ldots .$$

Proof: Assume that

$$y(t,\varepsilon) \leq v_{j-1}(t) .$$

Then

$$\varepsilon v_j'' = [(v_{j-1})^2 - t^2]v_j'$$

$$\varepsilon y'' = (y^2 - t^2)y' .$$

Hence

$$\varepsilon(v_j - y)'' = (v_{j-1}^2 - t^2)(v_j - y)' + (v_{j-1}^2 - y^2)y' .$$

Once more the maximum principle implies that

$$v_j - y \geq 0 .$$

Theorem 2.1: Let $A > B \geq 0$. There exists a solution $y_M(t,\varepsilon)$ which satisfies

2.10a)
$$y(t,\varepsilon) \leq y_M(t,\varepsilon)$$

for every solution $y(t,\varepsilon)$. Moreover, there is a solution $y_m(t,\varepsilon)$ which satisfies

2.10b)
$$y_m(t,\varepsilon) \leq y(t,\varepsilon)$$

for any solution $y(t,\varepsilon)$.

Remark: Of course, it may happen that

$$y_m = y_M .$$

Proof: Let

$$v_0(t) \equiv A .$$

Then,

$$v_1(t) \leq v_0(t) .$$

Thus

$$B \leq v_{j+1} \leq v_j, \qquad j = 0,1,2,...$$

and the functions $v_j(t)$ converge to a function $y_M(t,\varepsilon)$. Since the estimate (2.4) holds we also have

$$|v_j''(t)| \leq \frac{1}{\varepsilon} A^2 |A - B| \exp\left\{\frac{A^2 + 1}{\varepsilon}\right\} .$$

Thus, the first derivatives also converge. Finally, the differential equation implies that the second derivatives also converge. Thus, $y_M(t,\varepsilon)$ is a solution of (1.1), (1.2). The estimate (2.10a) follows from Lemma 2.2.

The proof of the existence of $y_m(t,\varepsilon)$ follows in the same way.

3. The Case $A = .96, B = .001$

In [4,5] G. Kedem considered the special problem (1.1), (1.2) when

3.1) $$A = .96, \quad B = .001, \quad \varepsilon = \frac{1}{15}.$$

Applying his theory (based on the Kantorovich Theorem and rigorous a-posteriori error bounds) he was able to prove that there are at least three solutions $y_{III}(t,\frac{1}{15}) \leq y_{II}(t,\frac{1}{15}) \leq y_I(t,\frac{1}{15})$ - see Figure 1. Furthermore these solutions satisfy

3.2a) $$|y_I'(-1,\frac{1}{15})| < A - B = .959,$$

3.2b) $$|y_I'(0,\frac{1}{15})| = \max|y_I'(t,\frac{1}{15})| > 1,$$

3.3a) $$|y_{III}'(-1,\frac{1}{15})| = \max|y_{III}'(t,\frac{1}{15})| > 1,$$

3.3b) $$|y_{III}'(0,\frac{1}{15})| < A - B = .959.$$

Moreover, $y_I''(t,\frac{1}{15})$, $y_{III}''(t,\frac{1}{15})$ have exactly one zero, where y_I and y_{III} cross y_0.

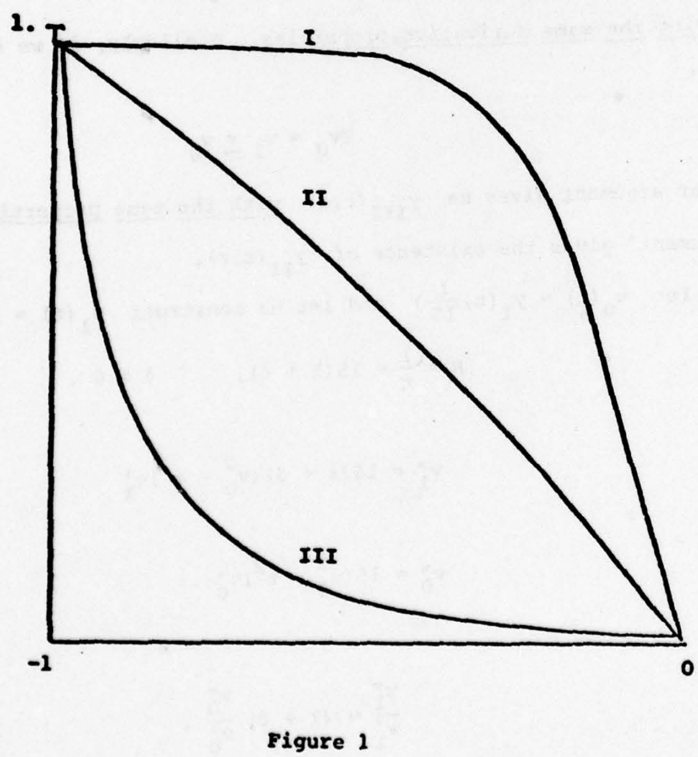


Figure 1

The purpose of this section is to prove the following result.

<u>Theorem 3.1</u>: Let $A = .96$, $B = .001$ and let $0 < \varepsilon < \frac{1}{15}$. Then, there are at least three solutions $y_I(t,\varepsilon)$, $y_{II}(t,\varepsilon)$, $y_{III}(t,\varepsilon)$ of (1.1), (1.2). Furthermore

3.4a) $$y_I(t,\tfrac{1}{15}) \le y_I(t,\varepsilon) \to A \quad \text{as} \quad \varepsilon \to 0+ ,$$

3.4b) $$y_{III}(t,\tfrac{1}{15}) \ge y_{III}(t,\varepsilon) \to B \quad \text{as} \quad \varepsilon \to 0+ .$$

In fact, if $0 < \varepsilon_1 < \varepsilon_2 \le \frac{1}{15}$ then

3.5) $$B \le y_{III}(t,\varepsilon_1) \le y_{III}(t,\varepsilon_2) \le y_I(t,\varepsilon_2) \le y_I(t,\varepsilon_1) \le A .$$

The proof follows from the following argument.

Let $v_0(t) = y_I(t,\frac{1}{15})$. Then using the properties of $y_I(t,\frac{1}{15})$, e.g. (3.2a), (3.2b), we see that

3.6a) $$Tv_0 = v_1 \ge v_0 .$$

Thus, applying Lemma 2.1 we obtain a sequence $v_j(t)$ that increases to a solution $y_I(t,\varepsilon)$ <u>having the same derivative properties</u>. Similarly, if we set $v_0(t) = y_{III}(t,\frac{1}{15})$ we find that

3.6b) $$Tv_0 = v_1 \le v_0$$

and a similar argument gives us $y_{III}(t,\varepsilon)$ <u>with the same properties</u>. Finally, a simple "degree argument" gives the existence of $y_{II}(t,\varepsilon)$.

Hence, let $v_0(t) = y_I(t,\frac{1}{15})$ and let us construct $v_1(t) = Tv_0$. Let

3.7) $$R = \frac{1}{\varepsilon} = 15(1+\delta), \qquad \delta > 0 .$$

Then

3.8a) $$v_1'' = 15(1+\delta)(v_0^2 - t^2)v_1'$$

while

3.8b) $$v_0'' = 15(v_0^2 - t^2)v_0' .$$

Thus

3.9a) $$\frac{v_1''}{v_1'} = (1+\delta)\frac{v_0''}{v_0'} .$$

That is

$$\text{3.9b)} \qquad |v_1'(t)| = C|v_0'(t)|^{1+\delta} .$$

To evaluate the constant $C = C(\delta)$ we integrate (3.9a) and obtain (using the boundary conditions (3.1))

$$\text{3.10a)} \qquad A - B = \int_{-1}^{0} |v_1'(t)|dt = C(\delta) \int_{-1}^{0} |v_0'(t)|^{1+\delta}dt ,$$

$$\text{3.10b)} \qquad C(\delta) = \frac{(A - B)}{\int_{-1}^{0} |v_0'(t)|^{1+\delta}dt} .$$

Lemma 3.1: The following inequalities hold.

$$\text{3.11a)} \qquad C(\delta)|v_0'(0)|^{\delta} \geq 1 ,$$

$$\text{3.11b)} \qquad C(\delta)|v_0'(-1)|^{\delta} \leq 1 .$$

Proof: Using (3.2b) we have

$$\frac{1}{C(\delta)} = \frac{\int_{-1}^{0} |v_0'(t)|^{1+\delta}dt}{A - B} \leq \frac{\int_{-1}^{0} |v_0'(t)|dt}{A - B} |v_0'(0)|^{\delta} .$$

Thus, (3.11a) follows from the boundary conditions which imply that

$$A - B = \int_{-1}^{0} |v_0'(t)|dt .$$

From Hölder's inequality we have

$$(A - B)^{1+\delta} = \left[\int_{-1}^{0} |v_0'(s)|ds\right]^{1+\delta} \leq \int_{-1}^{0} |v_0'(s)|^{1+\delta}ds .$$

Thus, using (3.2a) we have

$$|v_0'(-1)|^{\delta} < (A - B)^{\delta} \leq \frac{1}{C(\delta)}$$

which proves (3.11b).

In order to prove (3.6a) we consider the function

$$\text{3.12a)} \qquad F(t) = v_0(t) - v_1(t) .$$

Using (3.9b) we have

3.12b) $$F(t) = v_0(t) - A + C(\delta) \int_{-1}^{t} |v_0'(s)|^{1+\delta} ds .$$

We wish to show that $F(t) \leq 0$. We have

3.13) $$F(-1) = F(0) = 0 .$$

Moreover

$$F'(t) = v_0'(t) + C(\delta)|v_0'(t)|^{1+\delta} = |v_0'(t)|[C(\delta)|v_0'(t)|^{\delta} - 1] .$$

Thus, (3.11a) and (3.11b) imply that

$$F'(-1) < 0, \quad F'(0) > 0 .$$

Hence, if $F(t_0) > 0$ for some $t_0 \in (-1,0)$, then $F'(t)$ would have at least three zeros. If $F'(\bar{t}) = 0$, then

3.14) $$|v_0'(\bar{t})| = K(\delta) = [1/C(\delta)]^{\frac{1}{\delta}} .$$

However, if (3.14) has at least three solutions, then Rolle's theorem implies that $v_0''(t)$ has at least two zeros. But, $v_0''(t) = y_I''(t, \frac{1}{15})$ has exactly one zero. Thus, we have proven (3.2a). By the argument sketched earlier, we have obtained $y_I(t,\varepsilon)$. Moreover, $|y_I'(-1,\varepsilon)| < A - B$, $|y_I'(0,\varepsilon)| = \max|y'(t,\varepsilon)| > 1$ and $y_I''(t,\varepsilon)$ has exactly one zero.

In a similar way we obtain $y_{III}(t,\varepsilon)$.

In order to prove the existence of a third solution we follow the argument in [1], [9], or [6]. Roughly speaking, let.

$$\Omega \equiv \{y(t) \in C[-1,0],\ B \leq y \leq A\} ,$$

$$\Omega_1 \equiv \{y(t) \in C[-1,0],\ y_I(t, \tfrac{1}{15}) \leq y \leq A\} ,$$

$$\Omega_2 \equiv \{y(t) \in C[-1,0],\ B \leq y \leq y_{III}(t, \tfrac{1}{15})\} ,$$

$$\Omega_3 \equiv \Omega \backslash (\Omega_1 \cup \Omega_2) .$$

Then

$$T : \Omega \to \Omega$$

$$T : \Omega_1 \to \Omega_1$$

$$T : \Omega_2 \to \Omega_2$$

and, in each case, T maps the boundary into the interior. Thus, if $d(T,\tilde{\Omega})$ denotes the degree of T relative to the region $\tilde{\Omega}$ we have (see [1,8])

$$d(T,\Omega) = d(T,\Omega_1) = d(T,\Omega_2) = 1 .$$

Since the degree is additive,

$$d(T,\Omega_3) = -1$$

and there is a solution $y_{II}(t,\varepsilon) \in \Omega_3$.

In fact, the work of Amann [1] and Steuerwalt [9] shows that

$$y_{II}(t,\varepsilon) \not\geq y_I(t, \frac{1}{15}), \qquad y_{II}(t,\varepsilon) \not\leq y_{III}(t, \frac{1}{15}) .$$

These inequalities imply that any limit function $Y(t)$ of $y_{II}(t,\varepsilon)$ is truly distinct from $Y(t) \equiv A$ or $Y(t) \equiv B$.

Thus, we have proven Theorem 3.1.

4. $\lambda = 1,\ B = 0$

Turning to the special case where $A = 1$, $B = 0$ we see that

$$y_0(t,\varepsilon) = -t$$

is a solution of (1.1), (1.2) for all ε. Let

$$z(t,\varepsilon) = y(t,\varepsilon) + t ;$$

then if $y(t,\varepsilon)$ satisfies (1.1), (1.2) the function $z(t,\varepsilon)$ satisfies

4.1) $$\varepsilon z'' - z(z - 2t)z' + z(z - 2t) = 0, \qquad -1 \le t \le 0$$

4.2) $$z(-1,\varepsilon) = z(0,\varepsilon) = 0 .$$

Moreover, $z_0(t,\varepsilon) \equiv 0$ is a solution for all ε. In this situation it is natural to apply a bifurcation analysis. Linearizing (4.1), (4.2) about the solution $z_0(t,\varepsilon) \equiv 0$ we have the linear eigenvalue problem

4.3) $$\varphi'' + \lambda|t|\varphi = 0, \qquad -1 \le t \le 0$$

4.4) $$\varphi(-1) = \varphi(0) = 0$$

where

4.5) $$\lambda = \frac{2}{\varepsilon} .$$

The eigenvalues and eigenfunctions of this problem are easily obtained. In fact, let $\{j_{k,1/3}\}_{k=1}^{\infty}$ be the positive zeros of the Bessel function $J_{1/3}(x)$, ordered so that

$$j_{k,1/3} < j_{k+1,1/3} .$$

Let

4.6a) $$\mu_k = 3\left(\frac{j_{k,1/3}}{2}\right)^{\frac{2}{3}} ,$$

and

4.6b) $$\lambda_k = \frac{\mu_k^3}{3} = \frac{9}{4}\,(j_{k,1/3})^2 .$$

Then the λ_k, $k = 1,2,\ldots$ are the eigenvalues of (4.3), (4.4) while the corresponding eigenfunctions are given by

4.6c) $$\varphi_k(t) = \left(\frac{\mu_k|t|}{3}\right)^{\frac{1}{2}} J_{1/3}\left[2\left(\frac{|\mu_k t|}{3}\right)^{\frac{3}{2}}\right].$$

See [11, page 16].

The exact values of λ_k and $\varphi_k(t)$ are not of major importance here. The important fact is the following:

<u>Theorem 4.1</u>: Let $0 < \lambda_k < \frac{2}{\varepsilon}$. Then there are at least $2k + 1$ solutions of the nonlinear problem (4.1), (4.2). Moreover, a special set of $2k + 1$ solutions may be described in the following manner. Of course, we have the trivial solution,

4.7a) $$Z_0(t,\varepsilon) \equiv 0 .$$

Then there are $2k$ solutions $\{Z_j^+(t,\varepsilon)\},\{Z_j^-(t,\varepsilon)\}$, $j = 1,2,\ldots,k$ which satisfy

4.7b) $$\frac{d}{dt} Z_j^+(-1,\varepsilon) > 0, \quad \frac{d}{dt} Z_j^-(-1,\varepsilon) < 0$$

and $Z_j^{\pm}(t,\varepsilon)$ has exactly j interior nodal zeros. Moreover, $Z_j^+(t,\varepsilon)$ and $Z_j^-(t,\varepsilon)$ have the same zeros, say $t_1,t_2,\ldots,t_j$.

<u>Proof</u>: From the results of P. Rabinowitz [7] we find that there are two unbounded continua, C_j^+, C_j^-, in (Z,λ) space which meet at $(0,\lambda_j)$ and have no other points in common with the $(0,\lambda)$ line. Each point on these continua represents a solution of (4.1), (4.2) with exactly j interior zeros and the appropriate sign of $Z'(-1)$. Since the solutions $Z(t)$ of (4.1), (4.2) are bounded (via the maximum principle applied to (1.1), (1.2)) we see that these continua are bounded in Z-space and hence unbounded in λ. Thus, we have established the existence of $Z_j^{\pm}(t,\varepsilon)$ with interior nodal zeros $t_j^{\pm}$. However, if $y(t,\varepsilon)$ is a solution of (1.1), (1.2) then $y'(t,\varepsilon) < 0$. Thus we may consider its inverse function $g(y)$ defined by

$$g(y(t,\varepsilon)) = t .$$

Finally, let

$$G(y) = -g(y) .$$

A direct calculation now shows that: if $y(t,\varepsilon)$ is a solution of (1.1), (1.2) (with $A = 1$, $B = 0$) then so is $\tilde{G}(t,\varepsilon) = G(t)$. Thus, for every $Z^+(t,\varepsilon)$ there is a $Z^-(t,\varepsilon)$ with the same nodal zeros: Z^+ and Z^- are reflections of each other about Z_0.

Consider the following situation. Let j be fixed and let $\varepsilon_k \to 0+$ and (without loss of generality)

$$0 < \varepsilon_k < \frac{2}{\lambda_j} .$$

Let $\{z_j^{\pm}(t,\varepsilon_k)\}$ be a sequence of solutions of (4.1), (4.2). Let

4.8) $$-1 < t_1(\varepsilon_k) < t_2(\varepsilon_k) < \dots t_j(\varepsilon_k) < 0$$

be the zeros of $z_j^{\pm}(t,\varepsilon_k)$. Let

4.9) $$y_j^{\pm}(t,\varepsilon_k) = z_j(t,\varepsilon_k) - t$$

be the *corresponding solutions of* (1.1), (1.2). *Using the monotonicity of* $y_j^{\pm}(t,\varepsilon_k)$ and Helly's theorem [13] we know that we may choose subsequences $\varepsilon_{k'}$ - which we now call ε_k - so that

4.10a) $$t_\sigma(\varepsilon_k) \to \hat{t}_\sigma, \qquad \sigma = 1,2,\dots,j\ ,$$

4.10b) $$y_j^{\pm}(t,\varepsilon_k) \to Y^{\pm}(t), \text{ as } \varepsilon_k \to 0+\ .$$

Our first goal is to show that $Y^{\pm}(t)$ cannot coincide with the straight line $y = -t$ on any interval (α,β). The exact form of our results is a strengthening of the observations of [3].

Lemma 4.1: Let $-1 \le \alpha < \beta \le 0$. Let λ_0 be the smallest eigenvalue of the problem

4.11a) $$\varphi'' + \lambda|t|\varphi = 0, \qquad \alpha \le t \le \beta$$

4.11b) $$\varphi(\alpha) = \varphi(\beta) = 0\ .$$

Let $\varphi(t) = \varphi(t;\alpha,\beta)$ be the corresponding eigenfunction normalized so that

4.11c) $$\max \varphi(t;\alpha,\beta) = 1\ .$$

Suppose that

4.12a) $$0 < \varepsilon < \frac{1}{2}\varepsilon_0 = \frac{1}{4\lambda_0}\ .$$

Set

4.12b) $$\gamma_0 = \frac{1}{2\max|\varphi'(t)|}\ .$$

Let $y(t,\varepsilon)$ be a solution of (1.1) on the interval (α,β) which satisfies

4.13a) $$y(t,\varepsilon) + t > 0, \qquad \alpha < t < \beta\ ,$$

4.13b) $$y(\alpha,\varepsilon) + \alpha = y(\beta,\varepsilon) + \beta = 0\ .$$

Then

4.14) $$y(t,\varepsilon) \ge \gamma_0\varphi(t,\alpha,\beta) - t, \qquad \alpha \le t \le \beta\ .$$

Proof: Suppose the lemma is false. Then there is a function $y(t,\varepsilon)$ which satisfies the differential equation (1.1) and also satisfies (4.13a), (4.13b) but not (4.14). Nevertheless, there is a γ, $0 < \gamma \leq \gamma_0$, such that

$$w(t) = \gamma\varphi(t;\alpha,\beta) - t \leq y(t,\varepsilon)$$

and $w(t)$ "just touches" $y(t,\varepsilon)$. That is, either $w'(-1) = y'(-1,\varepsilon)$, or $w'(0) = y'(0,\varepsilon)$, or there is an interior point $t_0 \in (-1,0)$ such that

$$w(t_0) = y(t_0,\varepsilon) .$$

However, we will show that

4.15a) $$w_1 = Tw > w, \qquad \alpha < t < \beta$$

and

4.15b) $$w_1'(-1) > w'(-1)$$

4.15c) $$w_1'(0) < w'(0)$$

which, together with Lemma 2.2, contradicts the choice of γ as "just touching." To verify these facts we observe that

$$\varepsilon w'' = \gamma\varphi'' = -\frac{2\gamma\varepsilon}{\varepsilon_0}\varphi|t|$$

$$w^2 - t^2 = \gamma^2\varphi^2 - 2\gamma\varphi t$$

$$w' = \gamma\varphi' - 1 .$$

Thus

4.16a) $$\varepsilon w'' = (w^2 - t^2)w' + E$$

where

$$E = \gamma\varphi[-2\frac{\varepsilon}{\varepsilon_0}|t| - (2|t| + \gamma\varphi)(\gamma\varphi' - 1)] .$$

By (4.12a), (4.12b) and the choice of $\gamma < \gamma_0$ we see that $E(t) > 0$ for $\alpha < t < \beta$. Then, by the maximum principle we have (4.15a), (4.15b) and (4.15c).

Remark: The function $\varphi(t;\alpha,\beta)$, the eigenvalue λ_0, and the value γ_0 are all continuous functions of the pair α, β.

Lemma 4.2: Let $t_0 = -1$ and $t_{k+1} = 0$. Suppose (4.10a), (4.10b) hold. Let σ be fixed, $0 \le \sigma \le k$, and suppose that

4.17a) $$y_j^{\pm}(t,\varepsilon_k) + t > 0, \quad t_\sigma(\varepsilon_k) < t < t_{\sigma+1}(\varepsilon_k) .$$

We suppose that (4.10a), (4.10b) hold. Then

4.17b) $$Y_j^{\pm}(t) \equiv -\hat{t}_\sigma, \quad \hat{t}_\sigma \le t < \hat{t}_{\sigma+1} .$$

Similarly, if

4.18a) $$y_j^{\pm}(t,\varepsilon_k) + t < 0, \quad t_\sigma(\varepsilon_k) < t < t_{\sigma+1}(\varepsilon_k) ,$$

then

4.18b) $$Y_j^{\pm}(t) \equiv -\hat{t}_{\sigma+1}, \quad \hat{t}_\sigma < t \le \hat{t}_{\sigma+1} .$$

Proof: Consider the case when (4.17a) holds. If $\hat{t}_\sigma = \hat{t}_{\sigma+1}$ there is nothing to prove. Suppose then that

$$\hat{t}_{\sigma+1} - \hat{t}_\sigma = L > 0 .$$

As we remarked above, ε_0, $\varphi(t;\alpha,\beta)$ and γ_0 are continuous functions of (α,β). Hence for ε_k sufficiently small we have

4.19a) $$\varepsilon_k < \frac{1}{2}\varepsilon_0(\varepsilon_k) ,$$

4.19b) $$\frac{1}{2}\gamma_0(\hat{t}_\sigma,\hat{t}_{\sigma+1}) \le \gamma_0(t_\sigma(\varepsilon_k),t_{\sigma+1}(\varepsilon_k)) ,$$

and

4.19c) $$w(t,\varepsilon_k) = \frac{1}{2}\gamma_0\varphi(t,t_\sigma(\varepsilon_k),t_{\sigma+1}(\varepsilon_k)) - t \le y_j^{\pm}(t,\varepsilon_k) .$$

Let

4.20) $$y_j^{\pm}(t,\varepsilon_k) + t_\sigma(\varepsilon_k) = v(t)\exp\{-\frac{1}{\varepsilon}\int_t^{t_{\sigma+1}(\varepsilon_k)} (w^2(s,\varepsilon_k) - s^2)ds\} .$$

Then, the function $v(t)$ satisfies

4.21a) $$\varepsilon v'' - [2(w^2 - t^2) + (y^2 - t^2)]v' - \frac{(w^2 - t^2)}{\varepsilon}[y^2 - w^2]v = 0 ,$$

4.21b) $$v(t_\sigma(\varepsilon_k)) = 0 ,$$

4.21c) $$v(t_{\sigma+1}(\varepsilon_k)) = t_\sigma(\varepsilon_k) - t_{\sigma+1}(\varepsilon_k) .$$

Thus, the maximum principle implies that

$$|v(t)| \leq 1$$

and

$$|y^{\pm}(t,\varepsilon_k) + t_\sigma(\varepsilon_k)| \leq \exp\{-\frac{1}{\varepsilon}\int_t^{t_{\sigma+1}(\varepsilon_k)} [w^2(s,\varepsilon_k) - s^2]ds\} .$$

Thus, since $w(s,\varepsilon_k)$ is continuous in $(t_\sigma(\varepsilon_k), t_{\sigma+1}(\varepsilon_k))$ we obtain (4.17b).

A similar argument disposes of the case when (4.18a) is satisfied.

Thus, we see that when the $\hat{t}_\sigma$ are distinct $Y_j^+(t)$ is a step function with jumps at $\hat{t}_\sigma$ for odd values of σ. If σ is even, then

$$Y_j^+(t) \equiv -\hat{t}_\sigma, \qquad \hat{t}_{\sigma-1} < t < \hat{t}_{\sigma+1} .$$

Similarly, in the case of $Y_j^-(t)$, if the $\hat{t}_\sigma$ are distinct we see that $Y_j^-(t)$ is a step function with jumps at $\hat{t}_\sigma$ for even values of σ, and if σ is odd, then

$$Y_j^-(t) \equiv -\hat{t}_\sigma, \qquad \hat{t}_{\sigma-1} < t < \hat{t}_{\sigma+1} .$$

We now turn our attention to the determination of the values of $\hat{t}_\sigma$.

Consider the following situation. Let σ be fixed and consider the adjacent intervals $(t_{\sigma-1},t_\sigma)$, $(t_\sigma,t_{\sigma+1})$.

Case 1: $\qquad y_j^{\pm}(t,\varepsilon_k) + t > 0, \qquad t_{\sigma-1} < t < t_\sigma ,$

and

$$y_j^{\pm}(t,\varepsilon_k) + t < 0, \qquad t_\sigma < t < t_{\sigma+1}, \quad \text{(see Fig. 2)} .$$

Then

4.22a) $$\left|\frac{d}{dt} y_j^{\pm}(t_{\sigma-1},\varepsilon_k)\right| < 1 ,$$

and

4.22b) $$\left|\frac{d}{dt} y_j^{\pm}(t_{\sigma+1},\varepsilon_k)\right| < 1 .$$

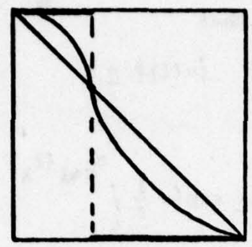


Figure 2

Integrating (1.1) we have $(y = y_j^{\pm}(t,\varepsilon_k))$

$$\varepsilon_k\left(\frac{d}{dt}y(t_{\sigma+1},\varepsilon_k) - \frac{d}{dt}y(t_{\sigma-1},\varepsilon_k)\right) = \frac{y^3}{3}\Bigg|_{t_{\sigma-1}}^{t_{\sigma+1}} - t^2y(t)\Bigg|_{t_{\sigma-1}}^{t_{\sigma+1}} + 2\int_{t_{\sigma-1}}^{t_\sigma} ty(t)dt$$

$$+ 2\int_{t_\sigma}^{t_{\sigma+1}} ty(t)dt\ .$$

Let $\varepsilon_k \to 0+$. Applying Lemma 4.2 we find that

4.23) $$\frac{\hat{t}_{\sigma-1}^3 - \hat{t}_{\sigma+1}^3}{3} = \hat{t}_\sigma^2(\hat{t}_{\sigma-1} - \hat{t}_{\sigma+1})\ .$$

If $\hat{t}_{\sigma-1} \neq \hat{t}_{\sigma+1}$ we have

4.24) $$\hat{t}_\sigma^2 = \frac{1}{3}(\hat{t}_{\sigma-1}^2 + \hat{t}_{\sigma-1}\hat{t}_{\sigma+1} + \hat{t}_{\sigma+1}^2)\ .$$

On the other hand, if $\hat{t}_{\sigma-1} = \hat{t}_{\sigma+1}$, then (4.24) holds also. Hence, (4.24) holds in the limit as $\varepsilon_k \to 0$ when starting from a triple $t_{\sigma-1}(\varepsilon_k)$, $t_\sigma(\varepsilon_k)$, $t_{\sigma+1}(\varepsilon_k)$ as in Case 1 above.

<u>Case 2:</u> $$y_j^{\pm}(t,\varepsilon_k) + t < 0, \quad t_{\sigma-1} < t < t_\sigma\ ,$$

$$y_j^{\pm}(t,\varepsilon_k) + t > 0, \quad t_\sigma < t < t_{\sigma+1}\ , \quad \text{(see Fig. 3)}\ .$$

It is not difficult to show that (with $y(t,\varepsilon) = y_j^{\pm}(t,\varepsilon)$)

4.25a) $$|y'(t_{\sigma-1},\varepsilon_k)| \le \frac{1}{\varepsilon_k}$$

4.25b) $$|y'(t_{\sigma+1},\varepsilon_k)| \le \frac{1}{\varepsilon_k}\ .$$

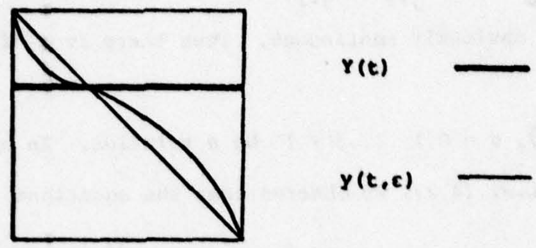


Figure 3

In this case we divide by $y'(t,\varepsilon)$ and obtain

$$\varepsilon_k\{\ln|y'(t_{\sigma+1},\varepsilon_k)| - \ln|y'(t_{\sigma-1},\varepsilon_k)|\} = \int_{t_{\sigma-1}}^{t_{\sigma+1}} (y^2 - t^2)dt .$$

Applying Lemma 4.2 we have

$$\hat{t}_\sigma^2(\hat{t}_{\sigma+1} - \hat{t}_{\sigma-1}) = \frac{1}{3}(\hat{t}_{\sigma+1}^3 - \hat{t}_{\sigma-1}^3) ,$$

and once more (4.24) holds. We summarize our results in

<u>Lemma 4.3</u>: If the values $t_\sigma(\varepsilon_k)$, $\sigma = 1,2,\ldots,j$ tend to limits $\hat{t}_\sigma$ as $\varepsilon_k \to 0$ then these limit values satisfy the quadratic equations

4.26)
$$\begin{cases} \hat{t}_\sigma^2 = \frac{1}{3}(\hat{t}_{\sigma-1}^2 + \hat{t}_{\sigma-1}\hat{t}_{\sigma+1} + \hat{t}_{\sigma+1}^2), & \sigma = 1,2,\ldots,j , \\ \hat{t}_0 = -1, \quad \hat{t}_{j+1} = 0 . \end{cases}$$

Finally, we show that (4.26) has one and only one solution, $-1 < \hat{t}_1 < \hat{t}_2 < \cdots \hat{t}_j < 0$.

<u>Lemma 4.4</u>: The system (4.26) has one and only one solution $\hat{t}_\sigma \in [-1,0]$. Moreover,

4.27)
$$-1 < \hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_j < 0 .$$

<u>Proof</u>: Consider the mapping $\hat{t}_\sigma = F_\sigma(\tau_0,\tau_1,\ldots,\tau_{j+1})$ given by

$$\begin{cases} \hat{t}_\sigma^2 = \frac{1}{3}(\tau_{\sigma-1}^2 + \tau_{\sigma-1}\tau_{\sigma+1} + \tau_{\sigma+1}^2), & \sigma = 1,2,\ldots,j , \\ \hat{t}_\sigma = -\sqrt{\hat{t}_\sigma^2} \end{cases}$$

where $\tau_0 = \hat{t}_0 = -1$, $\tau_{j+1} = \hat{t}_{j+1} = 0$. This mapping takes the cube, $-1 \le \tau_\sigma \le 0$ into itself and is obviously continuous. Thus there is a "fixed point" which is a solution of (4.26).

Let $\{\hat{t}_\sigma\}$, $\sigma = 0,1,\ldots,j+1$ be a solution. In order to verify the "separation" of the $\hat{t}_\sigma$, i.e. (4.27) we observe that the equations (4.26) imply that

$$\min(\hat{t}_{\sigma-1},\hat{t}_{\sigma+1}) \le \hat{t}_\sigma^2 \le \max(t_{\sigma-1}^2,t_{\sigma+1}^2) .$$

Suppose $\hat{t}_{\sigma-1}, \hat{t}_\sigma, \hat{t}_{\sigma+1} \le 0$ and $\hat{t}_\sigma = \hat{t}_{\sigma-1}$. Then $\hat{t}_{\sigma+1}$ satisfies the equation

$$\hat{t}_{\sigma+1}^2 + \hat{t}_\sigma \hat{t}_{\sigma+1} - 2\hat{t}_\sigma^2 = 0$$

i.e.

$$\hat{t}_{\sigma+1} = -\frac{\hat{t}_\sigma \pm 3\sqrt{\hat{t}_\sigma^2}}{2} .$$

Since $\hat{t}_\sigma \le 0$ we have

$$\hat{t}_{\sigma+1} = -\frac{\hat{t}_\sigma - 3\hat{t}_\sigma}{2} = \hat{t}_\sigma .$$

Thus, if two successive $\hat{t}_\sigma$ are equal, they are all equal. But that is impossible. Hence

$$-1 < \hat{t}_1 < \hat{t}_2$$

and a straightforward inductive argument proves (4.27).

Suppose $\hat{t}_\sigma$ and $\hat{s}_\sigma$ are both solutions of (4.26). Let

$$w_\sigma = \hat{t}_\sigma - \hat{s}_\sigma , \quad W_\sigma = \hat{t}_\sigma + \hat{s}_\sigma .$$

Then

4.28a) $$w_0 = 0, \quad w_j = 0 ,$$

and

4.28b) $$W_\sigma w_\sigma = \frac{1}{3}(W_{\sigma+1} + \frac{1}{2}W_{\sigma-1})w_{\sigma+1} + \frac{1}{3}(W_{\sigma-1} + \frac{1}{2}W_{\sigma+1})w_{\sigma-1} .$$

Let W be the tridiagonal matrix

$$[\frac{1}{3}(W_{\sigma+1} + \frac{1}{2}W_{\sigma-1}), -W_\sigma, \frac{1}{3}(W_{\sigma-1} + \frac{1}{2}W_{\sigma+1})] .$$

Clearly, W is irreducible. We will now show that

4.29) $$|w_\sigma| \geq \frac{1}{2}(|w_{\sigma+1}| + |w_{\sigma-1}|) .$$

Since $\hat{t}_\sigma$, $\hat{s}_\sigma$ are of the same sign, it suffices to show that (4.26) implies that

4.30) $$|\hat{t}_\sigma| \geq \frac{1}{2}(|\hat{t}_{\sigma+1}| + |\hat{t}_{\sigma-1}|) .$$

This estimate follows immediately from (4.26) and the well-known inequality

4.31) $$2\hat{t}_{\sigma-1}\hat{t}_{\sigma+1} \leq \hat{t}^2_{\sigma-1} + \hat{t}^2_{\sigma+1} .$$

Moreover, since the inequality in (4.31) is <u>strict</u> unless $\hat{t}_{\sigma-1} = \hat{t}_{\sigma+1}$ we see that, in fact,

$$|w_\sigma| > \frac{1}{2}(|w_{\sigma+1}| + |w_{\sigma-1}|)$$

and the matrix W is diagonally dominant. However, we have

$$Ww = 0 ,$$

thus $w = (w_\sigma) = 0$ (see [12]) and the lemma is proven.

Let us collect these results.

<u>Theorem 4.2</u>: For ε small enough, there exists $\{y_j^+(t,\varepsilon)\}$ a family of solutions of (1.1), (1.2) with exactly j interior turning points $t_\sigma(\varepsilon)$, i.e.

4.32a) $$y_j^+(t_\sigma(\varepsilon),\varepsilon) = -t_\sigma(\varepsilon), \qquad \sigma = 1,2,\ldots,j ,$$

and

4.32b) $$\frac{d}{dt} y_j^+(-1,\varepsilon) > -1 .$$

Then

4.33) $$t_\sigma(\varepsilon) \to \hat{t}_\sigma, \qquad \sigma = 1,2,\ldots,j ,$$

where the $\hat{t}_\sigma$ are the unique solutions of (4.26). Moreover, $y_j^+(t,\varepsilon) \to Y(t)$, a step function with jumps at $\hat{t}_\sigma$, σ odd, and

4.34) $$Y(t) = -\hat{t}_\sigma, \qquad \hat{t}_{\sigma-1} < t < \hat{t}_{\sigma+1}, \quad \sigma \text{ even} .$$

Also, for ε small enough, there exists $\{y_j^-(t,\varepsilon)\}$ a family of solutions with j interior turning points $t_\sigma(\varepsilon)$ and

4.35) $$\frac{d}{dt} y_j^-(-1,\varepsilon) < -1 ,$$

and $t_\sigma(\varepsilon) \to \hat{t}_\sigma$ as above. Moreover, $y(t,\varepsilon) \to Y(t)$, a step function with jumps at $\hat{t}_\sigma$, σ even, and

4.36) $$Y(t) = -\hat{t}_\sigma, \qquad \hat{t}_{\sigma-1} < t < \hat{t}_{\sigma+1}, \quad \sigma \text{ odd} .$$

<u>Remark:</u> The results of [3] are now seen to be a special case of the above results. The number $\frac{1}{\sqrt{3}}$ appears there in precisely the same way as the $\hat{t}_\sigma$ occur above.

5. $B = 0, \Lambda \neq 1$

Suppose we keep $B = 0$ and vary A. That is, let $A = A(x)$ be a continuous function of x with $A(0) = 1$. The question is, what happens to the solutions $y_j^{\pm}(t,\varepsilon)$? For example, when $0 < A < \frac{1}{\sqrt{3}}$ there cannot be a sequence $\varepsilon_n \to 0$ with

$$y(t,\varepsilon_n) \to \frac{1}{\sqrt{3}} .$$

Thus, that family is lost as A decreases. While we cannot give a complete discussion of the behavior of these solutions in terms of both variables A, ε, an almost complete answer is given by the following observation.

Suppose $\{y^{\pm}(t,\varepsilon)\}$ is a family of solutions of the differential equation (1.1) on the larger interval $[\beta(\varepsilon),0]$ with $\beta(\varepsilon) < -1$ which satisfy

(i)
$$y^{\pm}(\beta,\varepsilon) = -\beta \quad ,$$

(ii)
$$\frac{d}{dt} y^{+}(\beta,\varepsilon) > -1, \qquad \frac{d}{dt} y^{-}(\beta,\varepsilon) < -1 \ ,$$

and $y^{\pm}(t,\varepsilon)$ has exactly j turning points on the interval $\beta(\varepsilon) < t < 0$. (The existence of such solutions is guaranteed by the discussion of Section 4.) Suppose further that

$$y^{\pm}(-1,\varepsilon) = A \ .$$

Then, if all of the turning points actually lie in the smaller interval $(-1,0)$, we have obtained the solutions we seek. As we shall see, this is essentially the only way to obtain such solutions. This fact is the result of the following four theorems.

Theorem 5.1: Let $0 < A < 1$, $B = 0$. Let j be a fixed positive integer. Let $\hat{t}_2, \hat{t}_3, \ldots, \hat{t}_{j+1}$ be the solution of

5.1)
$$\hat{t}_k^2 = \frac{1}{3}(\hat{t}_{k-1}^2 + \hat{t}_{k-1}\hat{t}_{k+1} + \hat{t}_{k+1}^2), \qquad k = 2,3,\ldots,j \ ,$$

5.2)
$$\hat{t}_1 = -A, \qquad \hat{t}_{j+1} = 0 \ .$$

Let

5.3)
$$t_0 = -\frac{\hat{t}_2 - \sqrt{12A^2 - 3\hat{t}_2^2}}{2} \ .$$

Then, there exists an $\varepsilon_0 = \varepsilon_0(A)$ such that for all ε, $0 < \varepsilon \leq \varepsilon_0$, there is a solution $y(t,\varepsilon)$ of (1.1), (1.2) which has exactly j interior turning points, $-1 < t_1 < t_2 < \cdots < t_j < 0$, $(y(t_j,\varepsilon) = -t_j)$ and

5.4) $$0 > y'(-1,\varepsilon) > -\frac{1}{2}$$

if and only if

5.5) $$t_0 < -1 .$$

Moreover, if (5.5) holds then as $\varepsilon \to 0+$ we have

5.6) $$t_1(\varepsilon) \to -A = \hat{t}_1 ,$$

5.7) $$t_k(\varepsilon) \to \hat{t}_k, \qquad k = 2,3,\ldots,j ,$$

while $y(t,\varepsilon) \to Y(t)$, the step function given by

5.8) $$Y(t) = -\hat{t}_{2k+1}, \quad \hat{t}_{2k} < t < \hat{t}_{2k+2}, \qquad k = 0,1,2,\ldots,\bar{k} ,$$

where $\hat{t}_0 = -1$ and $\bar{k} = \frac{1}{2}(j - 1)$.

<u>Note</u>: If j is even, then $\bar{k}$ is a half integer and the last interval is actually $[\hat{t}_j,\hat{t}_{j+1}]$.

<u>Theorem 5.2</u>: Let $0 < A < 1$, $B = 0$. Let j be a fixed positive integer. Let $\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_j$ be the solution of

5.9) $$\hat{t}_k^2 = \frac{1}{3}(\hat{t}_{k-1}^2 + \hat{t}_{k-1}\hat{t}_{k+1} + \hat{t}_{k+1}^2), \qquad k = 1,2,\ldots,j ,$$

5.10) $$\hat{t}_0 = -1, \quad \hat{t}_{j+1} = 0 .$$

Then, there exists an $\varepsilon_0 = \varepsilon_0(A)$ such that for all ε, $0 < \varepsilon \le \varepsilon_0$ there is a solution $y(t,\varepsilon)$ of (1.1), (1.2) which has exactly j interior turning points $-1 < t_1 < \cdots < t_j < 0$ $(y(t_j,\varepsilon) = -t_j)$ and

5.11) $$y'(-1,\varepsilon) < -2$$

if and only if

5.12) $$-\hat{t}_1 < A .$$

Moreover, if (5.12) holds then as $\varepsilon \to 0+$ we have

5.13) $$t_k(\varepsilon) \to \hat{t}_k, \qquad k = 1,2,\ldots,j$$

while $y(t,\varepsilon) \to Y(t)$ where $Y(t)$ is the step function given by (5.8).

Theorem 5.3: Let $1 < A$, $B = 0$. Let j be a fixed positive integer. Let $\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_j$ be the solution of (5.9) with

5.14) $$\hat{t}_0 = -A, \quad \hat{t}_{j+1} = 0 .$$

Then, there is an $\varepsilon_0 = \varepsilon_0(A)$ such that for all ε, $0 < \varepsilon \leq \varepsilon_0$ there is a solution $y(t,\varepsilon)$ of (1.1), (1.2) which has exactly j interior turning points $t_k(\varepsilon)$, $k = 1,2,\ldots,j$ and

5.15) $$0 > y'(-1,\varepsilon) > -\frac{1}{2}$$

if and only if

5.16) $$-1 < \hat{t}_1 < 0 .$$

Moreover, if (5.16) holds then as $\varepsilon \to 0+$ we have

5.17) $$t_k(\varepsilon) \to \hat{t}_k, \qquad k = 1,2,\ldots,j$$

while $y(t,\varepsilon) \to Y(t)$, the step function given by

5.18a) $$Y(t) = A, \qquad -1 \leq t < \hat{t}_1$$

and

5.18b) $$Y(t) = -\hat{t}_{2k}, \qquad \hat{t}_{2k-1} < t < \hat{t}_{2k+1}, \qquad k = 1,2,\ldots,\bar{k}$$

where $\bar{k} = \frac{j}{2}$.

Theorem 5.4: Let $1 < A$, $B = 0$. Let j be a fixed positive integer. Let $\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_j$ be the solution of (5.9), (5.10). Let

5.19) $$\hat{t}_0 = -\frac{\hat{t}_1 - \sqrt{12 - 3\hat{t}_1^2}}{2} .$$

Then there exists an $\varepsilon_0 = \varepsilon_0(A)$ such that for all ε, $0 < \varepsilon \leq \varepsilon_0$, there is a solution $y(t,\varepsilon)$ which has exactly j interior turning points $t_1, t_2, \ldots, t_j$ and

5.20) $$y'(-1,\varepsilon) < -2$$

if and only if

5.21) $$-\hat{t}_0 > A .$$

Moreover, if (5.21) holds $t_k(\varepsilon) \to \hat{t}_k$, $k = 1,2,\ldots,j$ while $y(t,\varepsilon) \to Y(t)$, the step function given by (5.8).

The proofs of these theorems follow.

__Lemma 5.1__: Let $0 = B < A$. Let $y(t,\varepsilon_n)$, $\varepsilon_n \downarrow 0+$ be a sequence of solutions of (1.1), (1.2) which also satisfy (5.11) and have exactly j interior turning points $-1 < t_1(\varepsilon_n) < t_2(\varepsilon_n) < \cdots t_j(\varepsilon_n) < 0$. Then the points $t_k(\varepsilon_n) \to \hat{t}_k$, $k = 1,2,\ldots,j$ where the $\hat{t}_k$ satisfy (5.9), (5.10) and $y(t,\varepsilon) \to Y(t)$, the step function given by (5.8).

__Proof__: This lemma follows from the arguments of Section 4. The distinction is simply that in Section 4 we have Theorem 4.1 to assure the existence of certain special solutions.

__Lemma 5.2__: Let $0 = B < A$. Let $y(t,\varepsilon_n)$, $\varepsilon_n \downarrow 0+$ be a sequence of solutions of (1.1), (1.2) which also satisfy (5.4) and have exactly j interior turning points $-1 < t_1(\varepsilon) < \cdots < t_j(\varepsilon) < 0$. Then the points $t_k(\varepsilon_n) \to \hat{t}_k$, $k = 1,2,\ldots,j$. If $0 < A < 1$ the points $\hat{t}_k$ satisfy (5.1), (5.2) and if $A > 1$ the points $\hat{t}_k$ satisfy (5.9), (5.14). Moreover, the functions $y(t,\varepsilon_n)$ converge to the functions $Y(t)$ described in Theorem 5.1 and Theorem 5.3 respectively.

__Proof__: As in the lemma above, this lemma follows from the arguments of Section 4.

__Lemma 5.3__: $0 = B < A < 1$. Let $y(t,\varepsilon_n)$ be a sequence of solutions of (1.1), (1.2) which also satisfy the hypotheses of Lemma 5.2. Then $y(t,\varepsilon_n)$ may be continued backwards in $t < -1$ until $y(t,\varepsilon_n)$ crosses the curve $y \equiv -t$. Let this first turning point less than -1 be called $t_0(\varepsilon_n)$. Conceivably $t_0(\varepsilon_n) = -\infty$ for all ε_n. However, this is not the case. In fact, let $\hat{t}_k$ be the solutions of (5.1), (5.2) and let t_0 be given by (5.3); then

5.22) $$t_0(\varepsilon_n) \to t_0 .$$

__Proof__: Since $y'(t,\varepsilon_n) < 0$ the backward continuation of $y(t,\varepsilon_n)$ is above A for all $t < -1$. Therefore $y(t,\varepsilon)$ is bounded: $A \le y(t,\varepsilon) < -t$ as long as $y(t,\varepsilon)$ does not cross $y \equiv -t$. Thus $y(t,\varepsilon_n)$ may be continued backwards at least until such a crossing.

As $\varepsilon_n \to 0+$ we have

$$t_1(\varepsilon) \to \hat{t}_1 = -A, \quad t_2(\varepsilon) \to \hat{t}_2, \quad y'(t_2(\varepsilon_n),\varepsilon_n) < -2 .$$

Moreover,

$$y'(t_1(\epsilon_n),\epsilon_n) = y'(t_2(\epsilon_n),\epsilon_n)\exp\left\{\frac{1}{\epsilon}\int_{t_2(\epsilon_n)}^{t_1(\epsilon_n)}(y^2 - t^2)dt\right\}.$$

Since

$$\int_{t_2(\epsilon_n)}^{t_1(\epsilon_n)}(y^2 - t^2)dt \geq -(t_2(\epsilon_n) - t_1(\epsilon_n))^2$$

we have

5.23) $$|y'(t_2(\epsilon_n),\epsilon_n)| \geq 2\exp\{-\frac{1}{\epsilon}[t_2(\epsilon_n) - t_1(\epsilon_n)]^2\}.$$

Suppose $y(t,\epsilon_n)$ remains below $-t$ for $t < -1$. Then on any finite interval $\beta \leq t \leq t_1(\epsilon_n)$ we would have

$$y(t,\epsilon_n) \to A, \qquad \beta \leq t \leq \hat{t}_1,$$

$$y'(t,\epsilon_n) \to 0, \qquad \beta \leq t \leq \hat{t}_1.$$

However, in that case, if $\beta < -1$ we have

$$|y'(\beta,\epsilon_n)| \geq \lim 2\exp\{-\frac{1}{\epsilon_n}(t_2 - t_1)^2 + \frac{1}{\epsilon_n}(1 - A^2)|\beta|\}.$$

Thus, if $|\beta|$ is large enough

$$|y'(\beta,\epsilon_n)| \to +\infty.$$

Therefore, there must be a finite crossing $t_0(\epsilon_n)$. Moreover, this argument shows that for ϵ_n small enough

$$|t_0(\epsilon_n)| \leq \frac{2(\hat{t}_2 - \hat{t}_1)^2}{1 - A^2} + 1.$$

Let t_0 be any limit point of the $t_0(\epsilon_n)$. An argument similar to the arguments of Section 4 shows that the values $t_k(\epsilon)$, $k = 1,2,\dots,j$ must converge to the solutions of (5.9) with $\hat{t}_0 = t_0$. But the results of Lemma 5.2 determine the limits of the $t_k(\epsilon_n)$. Thus, t_0 must be given by (5.3).

<u>Corollary</u>: Since by construction, $t_0 < -1$, we have established one-half of Theorem 5.1.

Lemma 5.4: Suppose $0 = B < A < 1$ and (5.5) holds. For every δ, $0 < \delta < \frac{|t_0| - 1}{2}$, there exists an $\bar{\varepsilon} = \bar{\varepsilon}(\delta)$ such that for all ε, $0 < \varepsilon \le \bar{\varepsilon}$, there is a point $\tau = \tau(\varepsilon)$, $t_0 - \delta < \tau < t_0 + \delta$ and a function $y(t;\tau,\varepsilon)$ which satisfies

5.24a) $$\varepsilon y'' = (y^2 - t^2)y', \qquad \tau \le t \le 0 ,$$

5.24b) $$y(\tau;\tau,\varepsilon) = -\tau, \qquad y(0;\tau,\varepsilon) = 0 ,$$

5.24c) $$y(-1;\tau,\varepsilon) = A$$

and $y(t;\tau,\varepsilon)$ has exactly j turning points $t_k(\varepsilon)$, $k = 1,2,\ldots,j$ with $-1 < t_k(\varepsilon) < 0$.

Proof: For every τ, $t_0 - \delta < \tau < t_0 + \delta$, let $\tau_1, \tau_2, \ldots, \tau_j$ be the solutions of

5.25a) $$\tau_k^2 = \frac{1}{3}(\tau_{k-1}^2 + \tau_{k-1}\tau_{k+1} + \tau_{k+1}^2), \qquad k = 1,2,\ldots,j ,$$

5.25b) $$\tau_0 = \tau, \qquad \tau_{j+1} = 0 .$$

Since τ_1 is uniquely determined by τ_0 and τ_0 is uniquely determined by τ_1 when we consider the equations (5.25a) with τ_{j+1} given, we see that τ_1 is a monotone function of τ_0. Thus, if we choose $\tau_0 < t_0$ and let $y(t;\tau_0,\varepsilon)$ be a solution of (5.24a), (5.24b) with exactly j turning points and $y'(\tau_0;\tau_0,\varepsilon) < -1$, whose existence is guaranteed by (a simple modification of) Theorem 4.1, then for ε sufficiently small

$$y(-1;\tau_0,\varepsilon) < A \text{ and } -1 < t_1(\varepsilon) .$$

Similarly, if $t_0 < \tau_0$ we will obtain (for ε sufficiently small) a function $y(t;\tau_0,\varepsilon)$ which is a solution of (5.24a), (5.24b) having exactly j turning points and

$$y(-1;\tau_0,\varepsilon) > A, \qquad -1 < t_1(\varepsilon) .$$

Thus, there is an intermediate τ_0 which solves the problem.

Proof of Theorem 1: In the light of Lemma 5.2 and Lemma 5.3 it is only necessary to establish that (5.5) is a sufficient condition to guarantee that, for ε small enough, there is a solution of (1.1), (1.2) which has exactly j interior turning points. This result follows immediately from Lemma 5.4.

Proof of Theorem 5.2: The necessity of (5.12) follows immediately from Lemma 5.1 and the maximum principle, which implies that all solutions of (1.1), (1.2) satisfy

$$B \leq y(t,\varepsilon) \leq A .$$

Suppose (5.12) holds. Let

$$\hat{t}_{-1} = -\frac{\hat{t}_1 - \sqrt{12 - 3\hat{t}_1^2}}{2} < -1 . \tag{5.26}$$

The proof of Theorem 5.2 now follows precisely as in the proof of Lemma 5.4. We choose $\tau_0 < \hat{t}_{-1}$ and $\tau_0 > \hat{t}_{-1}$ and obtain the desired solutions which pass through $(-1,A)$ and are very steep at that point.

Proof of Theorem 5.3: If $y(t,\varepsilon)$ is a solution of (1.1), (1.2) which satisfies (5.15) then it certainly can be continued backward until it crosses $y \equiv -t$. The reason is that $y''(t,\varepsilon) < 0$ between $t = -1$ and any such crossing. Thus, moving backward $|y'(t,\varepsilon)|$ gets smaller and $y(t,\varepsilon)$ crosses $y \equiv -t$ at a value $t_{-1}(\varepsilon)$ which satisfies

$$-2A + 1 < t_{-1}(\varepsilon) < -1 .$$

Furthermore, as $\varepsilon \to 0$, $t_{-1}(\varepsilon) \to -A$. Looking at these solutions and applying the arguments of Section 4, we see that (5.16) must hold.

The sufficiency of (5.16) follows from the argument of Lemma 5.4.

The proof of Theorem 5.4 now follows the same lines as the proof of Theorem 5.3.

6. Remarks

The discussion of the case when $B \neq 0$ appears to be much more difficult. As long as $B = 0$ and we enlarge our basic interval by moving to the left we may continue to employ the theorem of Rabinowitz and general Sturm-Liouville theory. When we attempt to enlarge the interval to include some positive t the linearized equation (4.3) is no longer correct $(-t \neq |t|)$ and the general Sturm-Liouville theory becomes more delicate. We have not attempted a complete mathematical discussion of this case.

For this reason, the results of Section 3 and further computational results are particularly interesting. The graphs which follow are computational results for $\frac{1}{\varepsilon} = R = 150$ and the two sets of boundary conditions

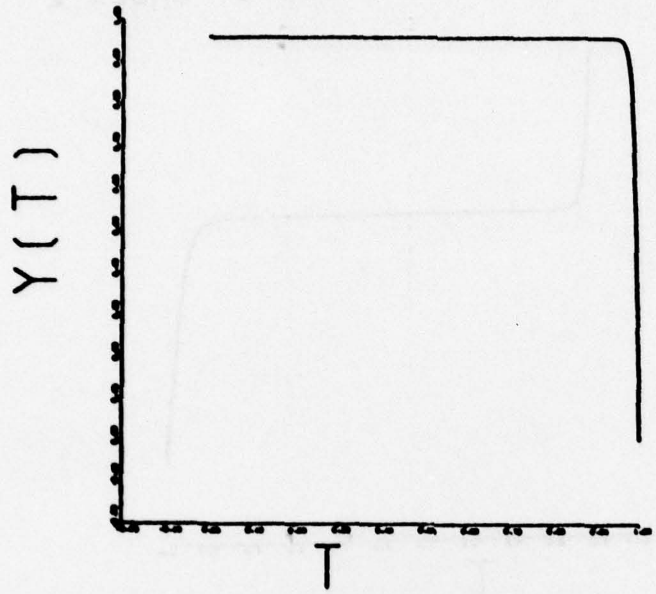
6.1) $$A = \Gamma, \quad B = 0 ,$$

6.2) $$A = .96, \quad B = .001 .$$

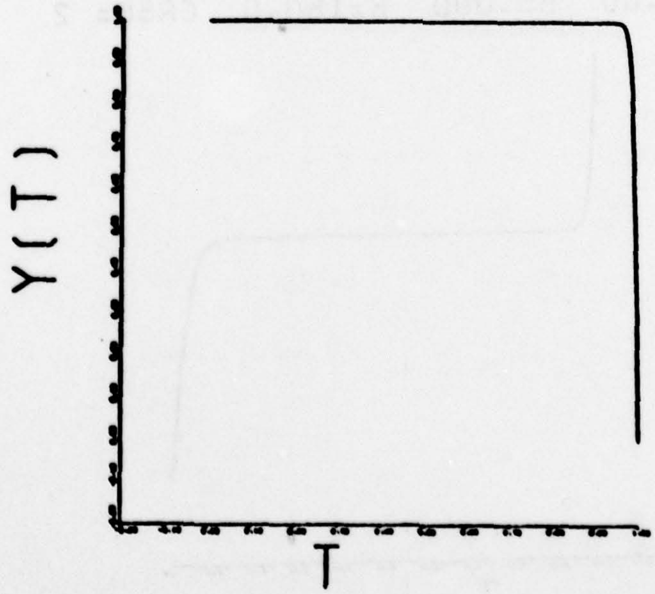
We find the "CASE 5" curves especially interesting.

These calculations were performed at the University of Rochester on the CDC 6600.

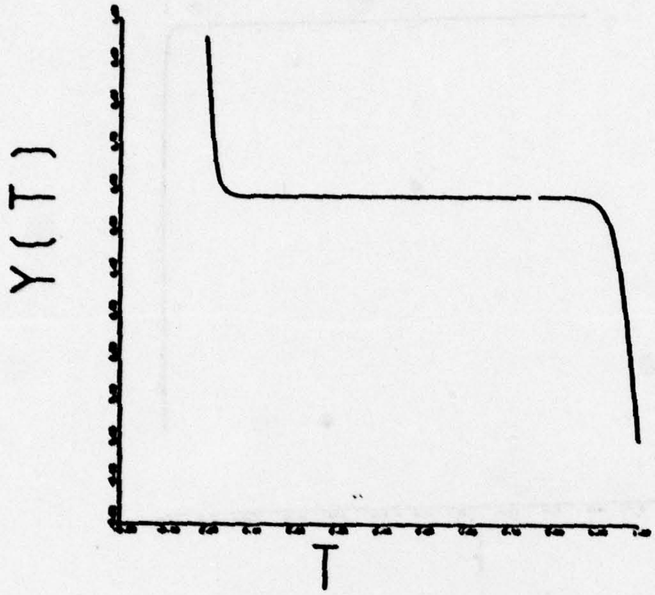
Y(T)'' = R*(Y(T)**2-(T-1)**2)*Y(T)'
A= .96 B=.001 R=150.0 CASE= 1



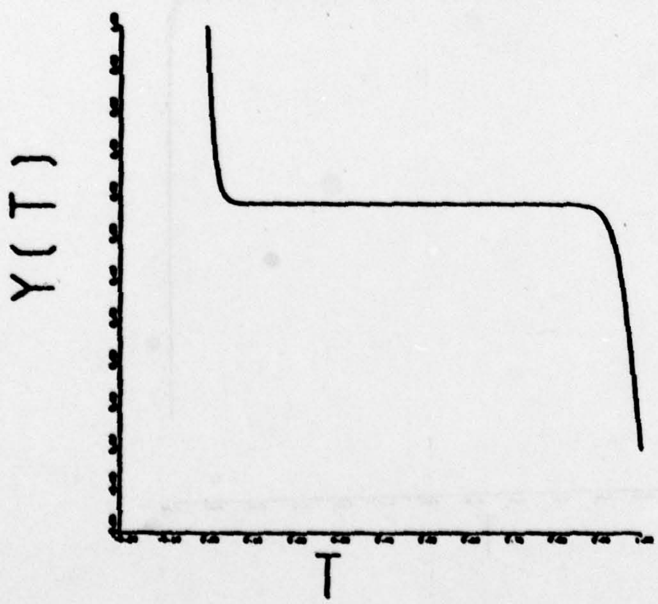
Y(T)'' = R*(Y(T)**2-(T-1)**2)*Y(T)'
A=1.00 B=.000 R=150.0 CASE= 1



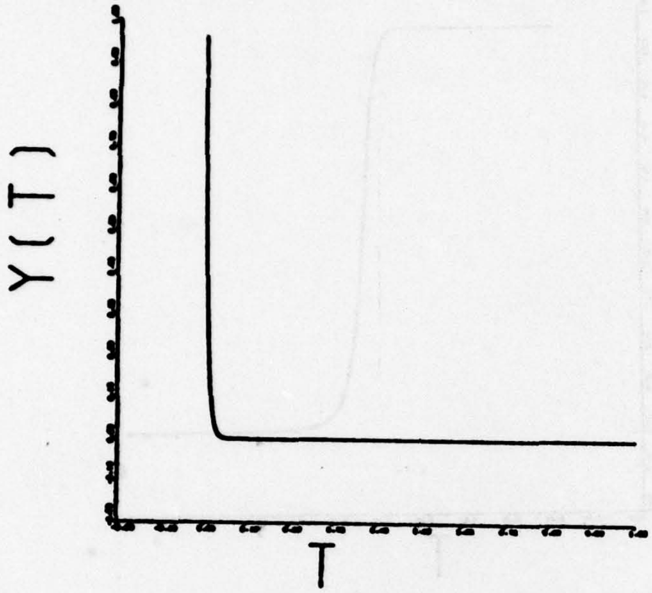
Y(T)'' = R*(Y(T)**2-(T-1)**2)*Y(T)'
A= .96 B=.001 R=150.0 CASE= 2



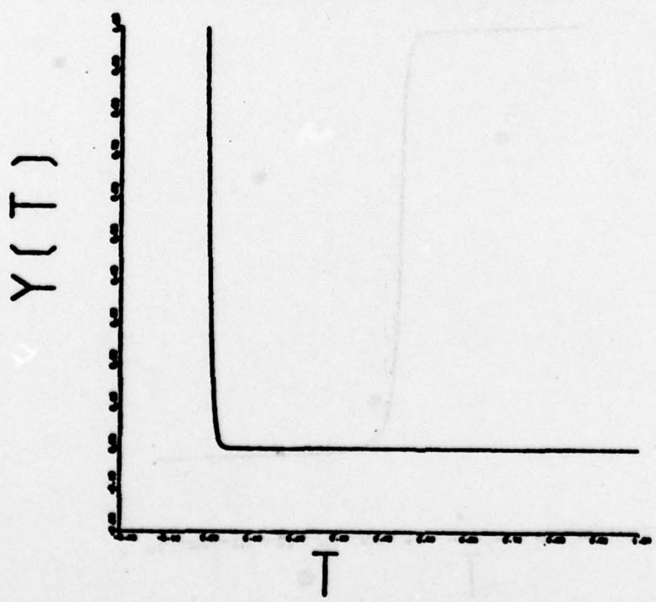
Y(T)'' = R*(Y(T)**2-(T-1)**2)*Y(T)'
A=1.00 B=.000 R=150.0 CASE= 2



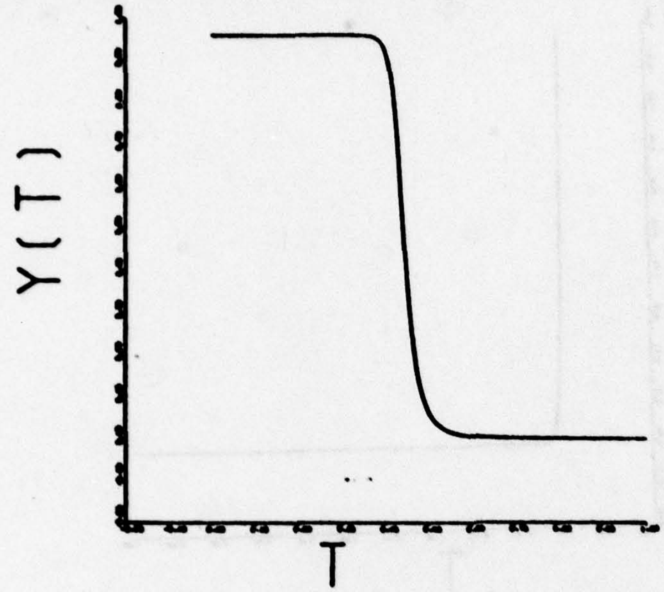
Y(T)'' = R*(Y(T)**2-(T-1)**2)*Y(T)'
A= .96 B=.001 R=150.0 CASE= 3



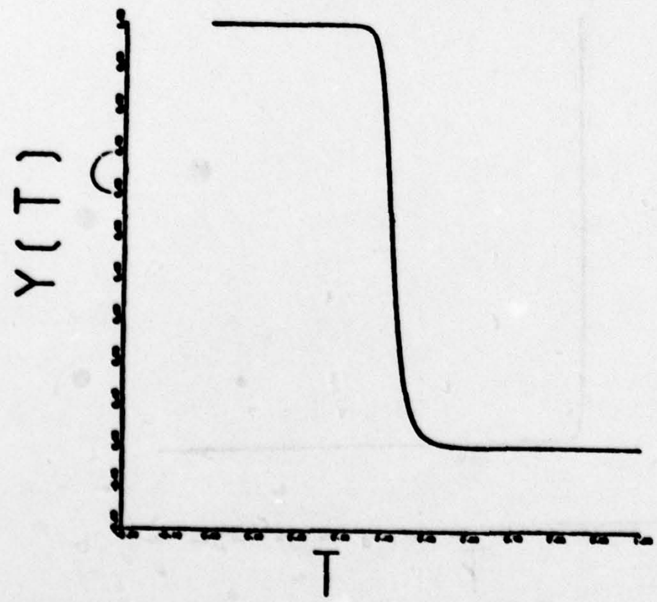
Y(T)'' = R*(Y(T)**2-(T-1)**2)*Y(T)'
A=1.00 B=.000 R=150.0 CASE= 3



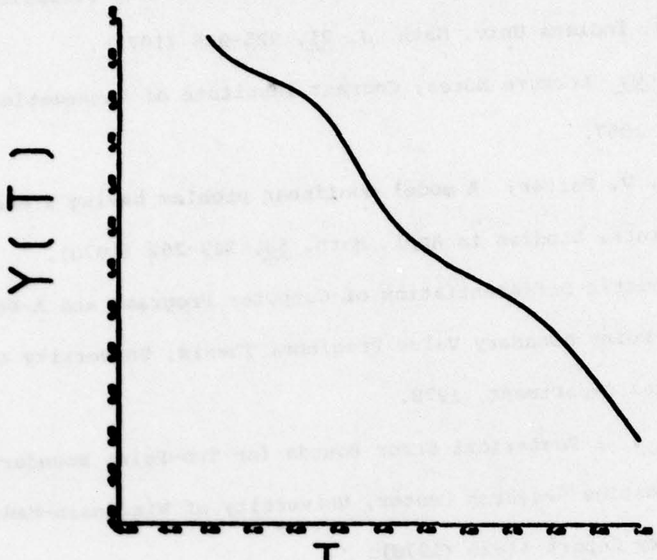
Y(T)'' = R*(Y(T)**2-(T-1)**2)*Y(T)'
A= .96 B=.001 R=150.0 CASE= 4



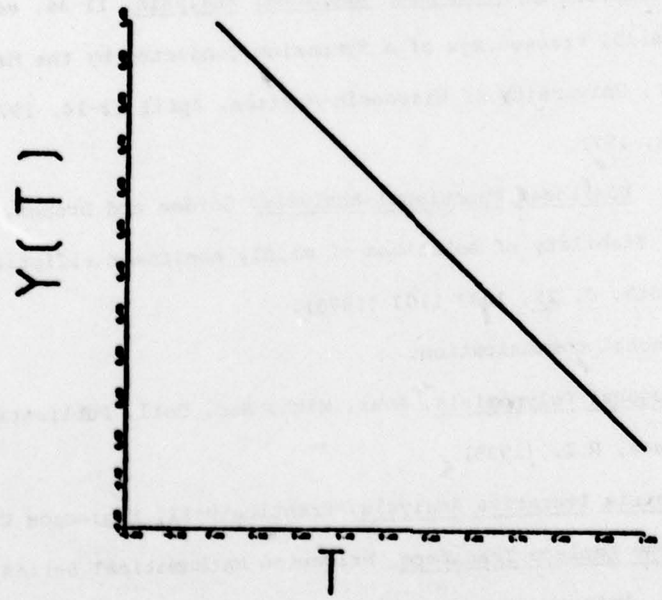
Y(T)'' = R*(Y(T)**2-(T-1)**2)*Y(T)'
A=1.00 B=.000 R=150.0 CASE= 4



Y(T)'' = R*(Y(T)**2-(T-1)**2)*Y(T)'
A= .96 B=.001 R=150.0 CASE= 5



Y(T)'' = R*(Y(T)**2-(T-1)**2)*Y(T)'
A=1.00 B=.000 R=150.0 CASE= 5



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